Lecture 3 (1/7/22)

More on Max Mod Principles.

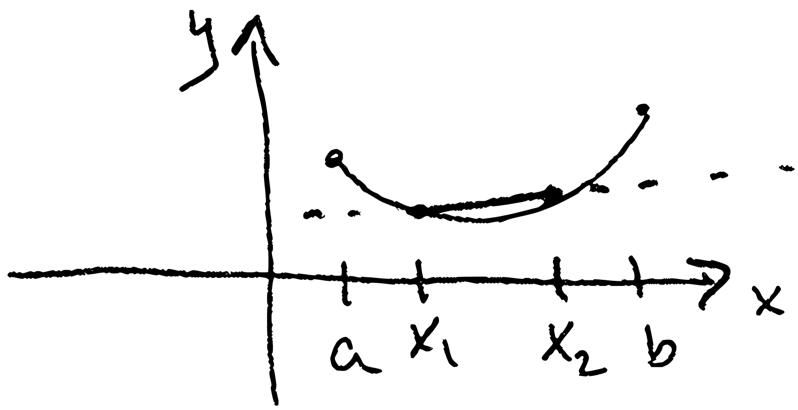
Convex functions.

We recall

Def. ① $u : [a,b] \to \mathbb{R}$ is <u>convex</u> if

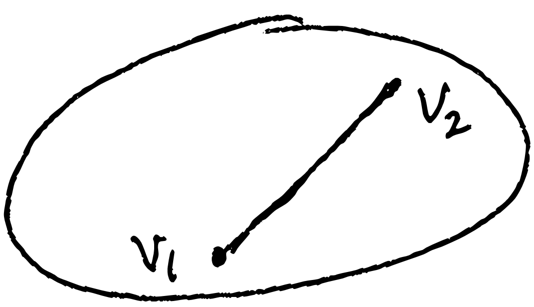
$$u(x_1(1-t) + x_2 t) \le (1-t)\,u(x_1) + t\,u(x_2)$$

for all $t \in [0,1]$ and $x_1, x_2 \in [a,b]$.



② A subset $A \subseteq \mathbb{R}^n$ is convex if

$$v_1, v_2 \subseteq A \Rightarrow [v_1, v_2] \subseteq A.$$



Note that $[v_1, v_2]$, as a path from v_1 to v_2 is parametrized by

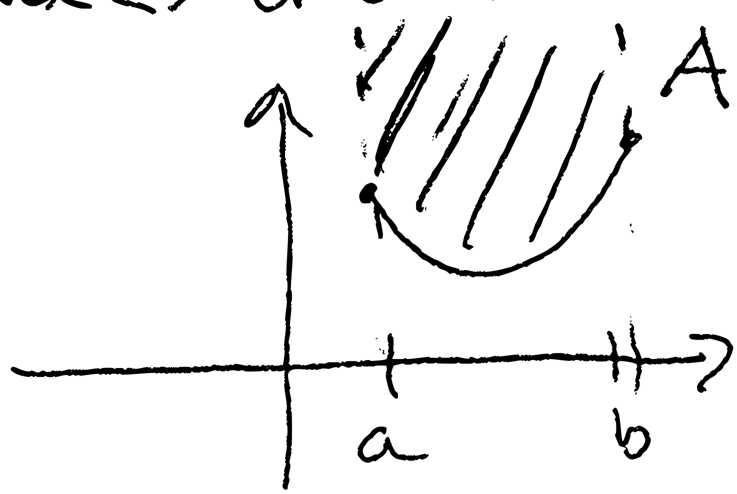
$$\gamma(t) = (1-t)v_1 + t v_2, \quad t \in [0,1].$$

<u>Prop1.</u> Let $u : [a,b] \to \mathbb{R}$ and

$$A = \{(x,y) \in \mathbb{R}^2 : x \in [a,b],\ y \geq u(x)\}.$$

Then, A convex $\Leftrightarrow$ u convex.

<u>Pf:</u> DIY.



<u>Prop2.</u> Let $u : [a,b] \to \mathbb{R}$.

(i) u convex $\Rightarrow$ u cont. on (a,b),

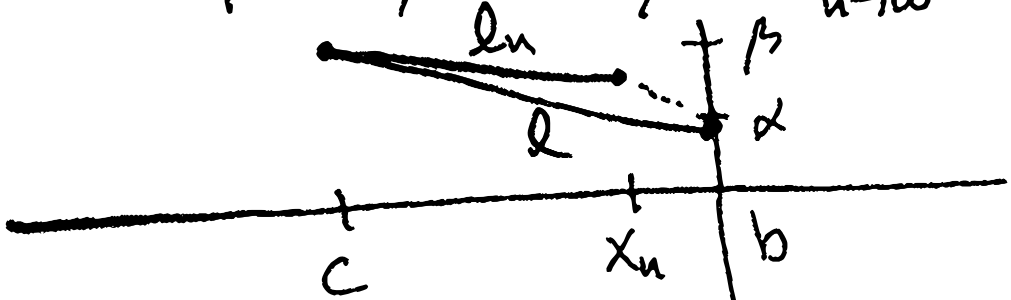
$$\lim_{x \to a^+} u(x) \leq u(a), \quad \lim_{x \to b^-} u(x) \leq u(b)$$

(ii) If u diff., then u convex $\Leftrightarrow$ u' increasing.

Pf (i) Let's show $\lim_{x\to b^-} u(x) \le u(b)$.

First, let $\alpha = \liminf_{x\to b} u(x) \le \beta = \limsup_{x\to b^-} u(x)$.

We first show $\beta \le \alpha \Rightarrow \beta = \alpha$. Pick any $c \in (a,b)$. Let $\{x_n\}$ be a sequence, $x_n \to b^-$, and $\lim_{n\to\infty} u(x_n) = \alpha$



Since any point $(x, u(x))$ for $x \in (c, x_n)$ must lie below or on the line l_n connecting $(c, u(c))$ to $(x_n, u(x_n))$, taking limit as $n \to \infty$, we conclude that any $(x, u(x))$, $x \in (c, b]$ must lie below or on the line l connecting $(c, u(c))$ to (b, α). Thus, if x_n' is another sequence, $x_n' \to b$, $u(x_n') \to \beta$, then $(x_n', u(x_n'))$ lies below or on $l \Rightarrow \beta \le \alpha \Rightarrow$ $\alpha = \beta \Rightarrow \lim_{x\to b^-} u(x)$ exists. A similar argument shows $\lim_{x\to b^-} u(x) \le u(b)$. Similar arguments work for limits from other side, interior points, and continuity. (DIY).

(ii) Pf is in Conway, Prop. 3.4. ☒

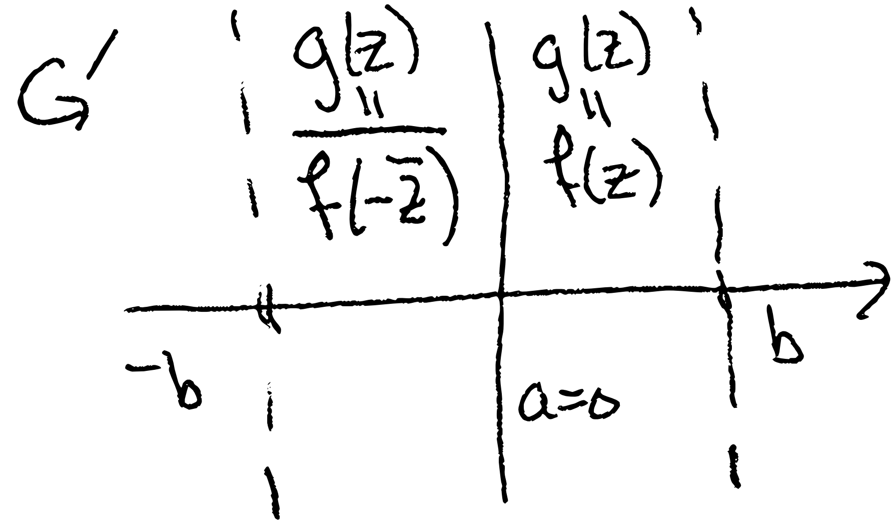
<u>Def.</u> ③ $u : [a,b] \to \mathbb{R}_{>0} = (0,\infty)$ is <u>log-convex</u> if $\log u$ is convex.

<u>Thm 1.</u> Let $G = \{ a < \operatorname{Re} z < b \}$ and f analytic in G, cont. in $\overline{G}$, and $|f| \le A$. Let $M(x) := \sup_{y \in \mathbb{R}} |f(x+iy)|$.

(i) If $M(x) = 0$ for $x \in [a,b]$, then $f \equiv 0$.

(2) If $M(x) > 0$, then $x \to M(x)$ is log-convex.

<u>Pf.</u> If $M(x) = 0$ for $x \in (a,b)$, then zeros of f in G have limit points in G $\Rightarrow$ $f \equiv 0$. Suppose $M(a) = 0$. WLOG: $a = 0$. Consider the function $g(z)$ in $G' = \{ -b < \operatorname{Re} z < b \}$



$f = 0$ on $\mathrm{Re}\, z = 0$

Clearly, g is cont. in $\overline{G'}$ and analytic in $G \cup \{-b < \mathrm{Re}\, z < 0\}$. An application of Morera's Thm (cf. HW in Fall) $\Rightarrow$ g is analytic in G'. Since $g(z) = 0$ on $\mathrm{Re}\, z = 0$, $g \equiv 0 \Rightarrow f \equiv 0$.

For pf of (ii), we need, notation as in Thm):

<u>**Lemma 1.**</u> Assume $M(a) \le 1$, $M(b) \le 1$, then $|f(z)| \le 1$.

<u>Rem.</u> Condition $|f| \le A$ in G is needed here by <u>Ex</u>: $b = -a = \pi/2$, $f(z) = e^{e^{iz}}$ (cf. Ex in Lecture 1).